EECE 380 Final Formula Sheet

Vector Formulas:

Cross Product Magnitude: $\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$

Dot Product: $\vec{A}\bullet\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$

Cartesian Coordinates: $x \rightarrow y \rightarrow z$

Cylindrical Coordinates: $r \rightarrow \varphi \rightarrow z$ $\qquad dl\,\varphi = r\,d\,\varphi$

Spherical Coordinates: $r \rightarrow \theta \rightarrow \phi$ $\qquad dl\,\theta = r\,d\,\theta\ ,\quad dl\,\phi = r\sin\theta d\,\phi$

Vector Projection of A on B: $\dfrac{\vec{A}\bullet\vec{B}}{\left|\vec{B}\right|^2}\vec{B}$

Normal Unit Vector to a Plane $ax + by + cz = cst$: $\dfrac{a\vec{x} + b\vec{y} + \vec{c}z}{\sqrt{a^2 + b^2 + c^2}}$

Gradient Properties: $\nabla\bullet(\nabla\times\vec{A}) = 0$ $\qquad \nabla\times(\nabla\vec{V}) = 0$

Stoke's Theorem: $\int_S (\nabla\times\vec{A})\bullet d\vec{s} = \oint_C \vec{A}\bullet d\vec{l}$

Divergence Theorem: $\int_V (\nabla\bullet\vec{A})dv = \oint_S \vec{A}\bullet d\vec{s}$

Electrostatics Formulas:

Electric Field Laws:

A) $\nabla \bullet \vec{E} = \frac{\rho_v}{\varepsilon_0}$

B) $\nabla \times \vec{E} = 0$

Stoke's Theorem for $\vec{E}$: $\int_S (\nabla \times \vec{E}) \bullet \vec{ds} = \oint_C \vec{E} \bullet \vec{dl} = 0$

Divergence Theorem for $\vec{E}$: $\int_V (\nabla \bullet \vec{E}) dv = \oint_S \vec{E} \bullet \vec{ds} = \frac{Q}{\varepsilon_0}$

Charge: $Q = \int_V \rho dv = \int_S \rho ds = \int_l \rho dl$

Field Created by a Point Charge: $\vec{E} = \frac{q}{4\pi\varepsilon_0 R^2} \vec{q}_r = \frac{q}{4\pi\varepsilon_0 |R_1 - R|^3} (\vec{R}_1 - \vec{R})$

Force between of one point charge on another: $\vec{F} = q\vec{E} = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2} \vec{a}_r$

Electric field created by a:

- Line Charge: $\vec{E} = \int_l \frac{\rho_l dl}{4\pi\varepsilon_0 R^2} \vec{a}_R$, Sheet: $\vec{E} = \int_S \frac{\rho_s ds}{4\pi\varepsilon_0 R^2} \vec{a}_R$, Volume: $\vec{E} = \int_V \frac{\rho_v dv}{4\pi\varepsilon_0 R^2} \vec{a}_R$
- Infinite Line: $\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0 r} \vec{a}_r$, Infinite Sheet: $\vec{E} = \frac{\rho s}{2\varepsilon_0} \vec{a}_n$, Ring: $\vec{E} = \frac{hb\rho_l}{2\varepsilon_0 (b^2 + h^2)^{3/2}} \vec{a}_n$
- Disk: $\vec{E} = \frac{Ah}{2\varepsilon_0} \left[\frac{-r}{\sqrt{r^2 + h^2}} + \ln\left(\frac{r + \sqrt{r^2 + h^2}}{h} \right) \right] \vec{a}_z$ (where A is the area)

Electric field at the center of a semi-circle in the x-y plane situated above the x-axis: $\vec{E} = -\frac{\rho_l}{2\pi\varepsilon_0 b} \vec{a}_y$

Gauss Law: $\nabla \bullet \vec{E} = \frac{\rho_v}{\varepsilon_0} \rightarrow \oint_S \vec{E} \bullet \vec{ds} = \frac{Q}{\varepsilon_0}$

Electric Potential:

$V = \frac{W}{q} = -\int \vec{E} \bullet \vec{dl}$, $V_A - V_B = -\int_A^B \vec{E} \bullet \vec{dl}$

$E = -\nabla V$

Electric Potential of a Point Charge: $V = \frac{q}{4\pi\varepsilon_0 R} = \frac{q}{4\pi\varepsilon_0 |\vec{r} - \vec{r}\,'|}$

Electric Potential for:

Line Charge: $V = \int_{\infty}^{A} \frac{\rho_l dl}{4\pi\varepsilon_0 |\vec{R} - \vec{R}\,'|}$, Sheet: $V = \int_{\infty}^{A} \frac{\rho_s ds}{4\pi\varepsilon_0 |\vec{R} - \vec{R}\,'|}$, Volume: $V = \int_{\infty}^{A} \frac{\rho_v dv}{4\pi\varepsilon_0 |\vec{R} - \vec{R}\,'|}$

Electric Potential at a point z above a charged disk: $V = \frac{\rho_s}{2\varepsilon_0}\left(\sqrt{r^2 + z^2} - |z|\right)$

Electric Dipole: $V_P = \frac{q}{4\pi\varepsilon_0 R^+} - \frac{q}{4\pi\varepsilon_0 R^-} \approx \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{p}{4\pi\varepsilon_0 R^2}\vec{a}_r$ where $\vec{p} = q\vec{d}$

Poisson Equation: $\nabla^2 V = \frac{-\rho_v}{\varepsilon_0}$

Dielectrics: $E = RJ \to J = \sigma E$

Density/Intensity Relation: $\vec{D} = \varepsilon\vec{E}$

Electric Susceptibility: $\nabla \bullet \vec{E} = \frac{\rho_v + \rho_{P_v}}{\varepsilon_0}$ $\quad \nabla \bullet \vec{P} = -\rho_{Pv}$ $\quad \varepsilon_r = 1 + \lambda_e$

$\vec{P} = \varepsilon_0 \lambda_e \vec{E}$ $\quad \vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0(1+\lambda_e)\vec{E} = \varepsilon_0\varepsilon_r\vec{E} = \varepsilon\vec{E}$

Boundary Conditions: $E_{2t} = E_{1t}$, $D_{1n} - D_{2n} = \rho_s$ or $\vec{a}_{21} \times \left(\vec{D}_1 - \vec{D}_2\right) = \rho_s$

$\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\varepsilon_2}{\varepsilon_1}$ (for a 0 charge density boundary) ,

Capacitance: $C = \frac{Q}{V}$

- Parallel Plate Capacitor: $C = \frac{\varepsilon S}{d}$
- Toroidal Capacitor: $C = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}$
- Spherical Capacitor: $C = \frac{4\pi\varepsilon}{\frac{1}{Ri} - \frac{1}{R_o}}$

Electric Energy: $W_e = \frac{1}{2}qV = \frac{1}{2}\int_V \vec{D} \bullet \vec{E} dv = \frac{1}{2}\int_V \varepsilon \left|\vec{E}\right|^2 dv$

Magnetostatics Formulas:

Magnetic Field Laws:

A) $\nabla \bullet \vec{B} = 0$

B) $\nabla \times \vec{B} = \mu_0 \vec{J}$

Stoke's Theorem for $\vec{B}$: $\int_S (\nabla \times \vec{B}) \bullet \vec{ds} = \oint_C \vec{B} \bullet \vec{dl} = \mu_0 I$

Divergence Theorem for $\vec{B}$: $\int_V (\nabla \bullet \vec{B}) dv = \oint_S \vec{B} \bullet \vec{ds} = 0$

Magnetic Field for:

- Infinite Line Current: $\vec{B} = \frac{\mu_0 I}{2\pi r} \vec{a}_\phi$
- The Inside of an Infinite Line Current: $\vec{B} = \frac{\mu_0 I_{cond} r}{2\pi b^2} \vec{a}_\phi$
- Toroidal Coil: $\vec{B} = \frac{\mu_0 NI}{2\pi r} \vec{a}_\phi$ (This value is 0 inside and outside the core)
- Finite Line Current: $\vec{B} = \frac{\mu_0 IL}{2\pi r \sqrt{L^2 + r^2}} \vec{a}_\phi$ (L is half the length of the line)
- Ring Current: $\vec{B} = \frac{\mu_0 I b^2}{2(b^2 + z^2)^{3/2}} \vec{a}_z$

Poisson Equation: $\nabla^2 A = -\mu_0 J$

Vector Potential: $\vec{A} = \frac{\mu_0 I}{4\pi} \oint_C \frac{\vec{dl}}{R}$

Magnetic Field/Flux Density: $\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 I}{4\pi} \oint_C \frac{\vec{dl} \times \vec{R}}{R^3}$ (Biot-Savart Law)

Magnetic Flux: $\Phi = \int_S \vec{B} \bullet \vec{ds} = \oint_C \vec{A} . \vec{dl}$

Force: $\vec{F} = I \oint \vec{dl} \times \vec{B}$ or $\vec{F} = q\vec{v} \times \vec{B}$

Magnetic Dipole Moment: $\vec{m} = IS\vec{a}_n$

Torque:

$T = \vec{d} \times \vec{F} = \vec{r} \times \vec{F}_1 + (-\vec{r}) \times \vec{F}_2$

$T = \vec{m} \times \vec{B}$

Magnetic Field Intensity of a Sheet: $\vec{H} = \frac{1}{2}\vec{J} \times \vec{a}_n$

Ampere's Law: $\oint_C \vec{H} \bullet \vec{dl} = I$

Density/Intensity Relation: $\vec{B} = \mu\vec{H}$

Magnetic Susceptibility: $\nabla \times \left(\frac{\vec{B}}{\mu_0} - \mu \right) = \vec{J}$ $\quad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ $\quad \mu_r = 1 + \lambda_m$

$\vec{M} = \lambda_m \vec{H}$ $\quad \vec{B} = \mu_0\vec{H} + \mu_0\vec{M} = \mu_0(1+\lambda_m)\vec{H} = \mu_0\mu_r\vec{H} = \mu\vec{H}$

Boundary Conditions: $B_{1n} = B_{2n}$, $H_{1t} - H_{2t} = J_s$ or $\vec{a}_{21} \times \left(\vec{H}_1 - \vec{H}_2 \right) = J_s$

$\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\mu_2}{\mu_1}$ (for a 0 charge density boundary)

Flux Linkage: $\psi = N\,\Phi$

Inductance: $L = \frac{\psi}{I}$

- Toroid: $L = \frac{\mu N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$
- Coaxial Transmission Line: $L = \frac{\mu I \ell}{2\pi} \ln\left(\frac{b}{a}\right)$
- Solenoid: $L = \frac{\mu N^2 S}{\ell}$

Magnetic Energy: $W_m = \frac{1}{2}qV = \frac{1}{2}\int_V \vec{B} \bullet \vec{H} dv = \frac{1}{2}\int_V \mu \left|\vec{H}\right|^2 dv$

Time Varying Fields:

Faraday's Law: $\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \quad \rightarrow \quad V = -\frac{Nd\Phi}{dt} \quad \Leftrightarrow \quad V = \int (\vec{u} \times \vec{B}) \bullet d\vec{l}$ (Latter used for moving circuits)

Note: $\vec{u} \times \vec{B}$ always points in the direction of the induced current; integral is set up opposite to direction of current.

Maxwell's Contribution: $\nabla \times \vec{B} = \mu_0 J + \mu_0 \frac{\partial D}{\partial t}$

Moving Circuit with Time Varying Field: $V \simeq E = -\int_S \frac{\partial B}{\partial t} + \oint_C (\vec{u} \times \vec{B}) \bullet d\vec{l}$, $\vec{u} = \vec{\omega} R$, $\alpha = \omega t$

Displacement and Conduction Current: $\nabla \times \vec{H} = J_C + J_D = J_C + \frac{\partial D}{\partial t}$ and $\frac{J_C}{J_D} = \frac{\sigma}{\omega\varepsilon}$

Poisson Equation: $\nabla^2 A = -\mu J + \mu\varepsilon \frac{\partial^2 A}{\partial t^2}$

Plane Wave Propagation:

Wave Equation: $A \cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right)$

Relations: $f = \frac{1}{T}$, $\omega = 2\pi f = \frac{2\pi}{T}$, $u_p = \frac{\lambda}{T} = \lambda f$, $B = \frac{2\pi}{\lambda}$

T: Timelength/Period (s)
λ: Spatial/Wave length (m)
f: Frequency (Hz)
ω: Angular Frequency (rad/s)
B: Wave Number or Phase Constant (rad/m)

Time Harmonic Fields:

$$\nabla \bullet \tilde{E} = \frac{\tilde{\rho}_V}{\varepsilon}$$
$$\nabla \times \tilde{E} = -j\omega\tilde{B} = -j\omega\mu\tilde{H}$$
$$\nabla \bullet \tilde{H} = \nabla \bullet \tilde{B} = 0$$
$$\nabla \times \tilde{H} = \tilde{J} + j\omega\tilde{D} = \tilde{J} + j\omega\varepsilon\tilde{E}$$

Complex Permeability: $\nabla \times \tilde{H} = j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\tilde{E} = j\omega\varepsilon_C\tilde{E}$, $\varepsilon_C = \left(\varepsilon - \frac{j\sigma}{\omega}\right)$, $\varepsilon_{real} = \varepsilon$, $\varepsilon_{imaginary} = \frac{\sigma}{\omega}$

Maxwell's Equations in a Charge-Free Medium:

$$\nabla \bullet \tilde{E} = 0$$
$$\nabla \times \tilde{E} = -j\omega\tilde{B} = -j\omega\mu\tilde{H}$$
$$\nabla \bullet \tilde{H} = \nabla \bullet \tilde{B} = 0$$
$$\nabla \times \tilde{H} = j\omega\varepsilon_C\tilde{E}$$

$$\nabla^2\tilde{E} + \omega^2\mu\varepsilon_C\tilde{E} = 0$$
$$\nabla^2\tilde{E} - \gamma^2\tilde{E} = 0$$
$$\nabla^2\tilde{E} + k^2\tilde{E} = 0$$

γ : Propagation Constant
k: Wavenumber

$$\gamma^2 = -\omega^2\mu\varepsilon_C \ , \ k^2 = \omega^2\mu\varepsilon_C$$
$$\gamma = j\,\omega\sqrt{\mu\varepsilon_C} \ , \ k = \omega\sqrt{\mu\varepsilon_C}$$

Solving:

The wave will not have a component in the direction of propagation:

$$\tilde{E}_x(z) = E_{x_0}^+ e^{-jkz} + E_{x_0}^- e^{+jkz}$$
$$\tilde{E}_y(z) = E_{y_0}^+ e^{-jkz} + E_{y_0}^- e^{+jkz}$$
$$\tilde{E}_z(z) = 0$$

Assuming no y component and only positive E_x component:

$$\tilde{H}_x = 0 \ , \tilde{H}_z = 0$$

Intrinsic Impedance: $\eta = \dfrac{\omega\mu}{k} = \sqrt{\dfrac{\mu}{\varepsilon}}$

$$\tilde{H}_y = -\frac{1}{jw\,\mu}\frac{\partial\tilde{E}_x(z)}{\partial z} = -\frac{1}{jw\,\mu}E_{x_0}^+(-jk)e^{-jkz} = \tilde{H}_{y_o}^+ e^{-jkz} \quad \text{where } \tilde{H}_{y_o}^+ = \frac{k}{w\,\mu}E_{x_0}^+ = \frac{E_{x_0}^+}{\eta}$$

Wave Relations in Electromagnetics:

$$u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \ , \text{ In free space: } u_p = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c = 3\times10^8$$

Relation Between E,H, and Direction of Propagation:

$$\tilde{H} = \frac{1}{\eta}\vec{a}_k \times \tilde{E} \quad \text{or} \quad \tilde{E} = -\eta\vec{a}_k \times \tilde{H}$$

In General:

$$\tilde{E} = \vec{a}_x E_x^+(z) \mp \vec{a}_y E_y^+(z)$$
$$\tilde{H} = \vec{a}_x H_x^+(z) \mp \vec{a}_y H_y^+(z)$$

so: $H_x^+(z) = \dfrac{-E_y^+(z)}{\eta}$ and $H_y^+(z) = \dfrac{-E_x^+(z)}{\eta}$

Wave Polarization:

Let $\tilde{E}(z) = \vec{a}_x\tilde{E}_x + \vec{a}_y\tilde{E}_y$ with

$$\tilde{E}_x(z) = E_{x_0}^+ e^{-jkz} = \left|E_{x_0}^+\right| e^{-jkz} e^{+j\phi x}$$
$$\tilde{E}_y(z) = E_{y_0}^+ e^{-jkz} = \left|E_{y_0}^+\right| e^{-jkz} e^{+j\phi y}$$

With E_x as reference: $E_{x_0}^+ = A_x$ and $E_{y_0} = \left|E_{y_0}^+\right| e^{j\delta} = A_y e^{j\delta}$

$$E(z,t) = \vec{a}_x A_x \cos(\omega t - kz) + \vec{a}_y A_y \cos(\omega t - kz + \delta)$$

$$|E(z,t)| = \sqrt{\left(A_x \cos(\omega t - kz)\right)^2 + \left(A_y \cos(\omega t - kz + \delta)\right)^2}$$

$$\Psi(z,t) = \tan^{-1}\left(\frac{A_y \cos(\omega t - kz + \delta)}{A_x \cos(\omega t - kz)}\right)$$ (usually z is taken equal to 0)

- For Linear Polarization (z taken equal to 0):

$\delta = 0$ or $\delta = \pi$

If $\delta = 0$: $\Psi(0,t) = \tan^{-1}\left(\frac{A_y \cos(\omega t)}{A_x \cos(\omega t)}\right) = \frac{A_y}{A_x}$ (constant)

If $\delta = \pi$: $\Psi(0,t) = \tan^{-1}\left(\frac{A_y \cos(\omega t + \pi)}{A_x \cos(\omega t)}\right) = -\frac{A_y}{A_x}$ (constant)

In both cases: $|E(0,t)| = \cos t(\omega t)\sqrt{A_x{}^2 + A_y{}^2}$

- For Circular Polarization (z taken equal to 0 and $A_x=A_y=A$):

$\delta = \frac{\pi}{2}$ or $\delta = -\frac{\pi}{2}$

If $\delta = \frac{\pi}{2}$ Left Hand Circular Polarization: $\Psi(0,t) = \tan^{-1}\left(-\frac{A \sin(\omega t)}{A \cos(\omega t)}\right) = -\omega t$

If $\delta = -\frac{\pi}{2}$ Right Hand Circular Polarization: $\Psi(0,t) = \tan^{-1}\left(\frac{A \sin(\omega t)}{A \cos(\omega t)}\right) = \omega t$

In both cases: $|E(0,t)| = A\sqrt{\cos^2(\omega t) + \sin^2(\omega t \pm \frac{\pi}{2})} = A$ (constant)

- For Elliptical Polarization:

δ is neither of the previous angles, both the magnitude and angle are not constant.

If $\sin\delta > 0$ left hand elliptical polarization.

If $\sin\delta < 0$ right hand elliptical polarization.

Constants:

$$\varepsilon_0 = \frac{10^{-9}}{36\pi} (F / m)$$

$$\mu_0 = 4\pi \times 10^{-7} (H / m)$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi(\Omega) \text{ (in free space)}$$

$$c = 3 \times 10^8 (m / s)$$